DIFFEOMORPHISMS OF C^r AND ANALYTIC MANIFOLDS

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1. Statement of the Theorem

We aim to prove Theorem 1 from Whitney's paper [W] while updating the terminology and attempting to explain each step in greater detail. This theorem is stated as follows

Theorem 1.1. *Any C^r-m-manifold ($r \geq 1$ finite or infinite) is C^r-diffeomorphic with an analytic manifold in Euclidean space $\mathbb{R}^{2m+1}$.*

This is done in two parts. First, we show that any C^r-m-manifold ($r \geq 1$ finite or infinite) is C^r-diffeomorphic with a manifold in Euclidean space $\mathbb{R}^{2m+1}$. The details of this can be found in [W], as we will mainly be focusing on the second half. Then, we will add the analyticity condition to complete the theorem.

Before we begin, we need to define some terminology.

2. Definitions

We begin with a short overview of topological manifolds. The definitions that follow are adapted from [L], which is a great place to start if you need a more comprehensive discussion of manifolds.

Definition 2.1. A topological space M is an n-manifold if

(1) M is a Hausdorff space
(2) M is second countable: There is a countable basis for the topology of M
(3) M is locally Euclidean with dimension n: Every point of M has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. The pair (U, ϕ) where U is the aforementioned open neighborhood and ϕ is the homeomorphism is known as a chart. The collection of all these charts covering M is called an atlas.

This allows us to think of a manifold as a patchwork collection of copies of $\mathbb{R}^n$, in each of which we can apply many of the mathematical theorems for Euclidean space. In general, we examine functions and their properties by looking at their image under charts.

Definition 2.2. Two charts (U, ϕ) and (V, σ) are C^r-compatible if either $U \cap V = \emptyset$ or the appropriate restriction of $\sigma \circ \phi^{-1}$ is a C^r-diffeomorphism.

Definition 2.3. Let M be an n-manifold

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- An atlas A on M is maximal if any chart that is C^r-compatible with a chart in A is already in A. We call such a maximal atlas a C^r-structure on M.
- M is a C^r-n-manifold if there exists a C^r-structure on M.

This allows us to look at functions defined over multiple charts without its definition "breaking" in the overlap of charts.

Definition 2.4. Let M, N be C^r-n-manifolds. $f : M \to N$ is a C^r-map if for every $p \in M$ there exist smooth charts (U, ϕ) containing p and (V, σ) containing $f(p)$ such that $f(U) \subseteq V$ and the map $\sigma \circ f \circ \phi^{-1} : \phi(U) \to \sigma(V)$ is C^r.

Definition 2.5. Let M, N be C^r-n-manifolds. $f : M \to N$ is a C^r-diffeomorphism if it is a C^r-map and has a C^r-inverse. If such a map exists, we say that M and N are C^r-diffeomorphic.

Definition 2.6. Let M be a C^r-m-manifold and N be a C^r-n-manifold. A C^r-map $f : M \to N$ is a C^r-immersion if its differential is injective at each point of M.

Definition 2.7. Let M be a C^r-m-manifold and N be a C^r-n-manifold. A C^r-embedding of M into N is a C^r-immersion $f : M \to N$ that is also a homeomorphism onto its image in the subspace topology.

3. The Embedding Theorem and Assumed Lemmas

The following two lemmas are extensions of the results in another of Whitney's papers, [AE].

Lemma 3.1. *Let f be a C^r-map $(r \geq 0)$ of the open set $R \subseteq \mathbb{R}^m$ into $\mathbb{R}^n$, and $\eta : R \to \mathbb{R}$ be positive and continuous. Then there is an analytic map $F(p)$ on R which approximates f in R within η through the rth order.*

Lemma 3.2. *Let A be a closed subset of the C^r-manifold M. Let $\eta : M \backslash A \to \mathbb{R}$ be positive and continuous on $M \backslash A$, let $\eta(p) \to 0$ as p approaches any point of A, and let $f : M \to N$ be a C^r-map into the C^r-manifold N. If F approximates f in $M \backslash A$ within η to the rth order and $F = f$ in A, then F is of class C^r on M.*

The following lemma defines a neighborhood of a manifold embedded in Euclidean space. It will be used extensively when proving the theorem.

Lemma 3.3. *Let M be a C^r-m-manifold embedded in $\mathbb{R}^n$ ($r \geq 1$ finite or infinite). Then there is a positive continuous function $\xi : M \to \mathbb{R}$ and a function P of class C^r on M such that:*

1. *$P(p)$ is an $(n-m)$-plane through p complementary to the tangent plane to M at p.*
2. *If $R(p)$ is that part of $P(p)$ within $\xi(p)$ of p, then the $R(p)$ fill out a neighborhood $R(M)$ of M in a $(1-1)$ way. That is, if a point $q \in R(M)$ satisfies $q \in R(p_1)$ and $q \in R(p_2)$, then $p_1 = p_2$.*
3. *If $H(q) = p$ for $q \in R(p)$, then H is of class C^r on $R(M)$. Moreover, if M is analytic, so are P and H.*

4. The Analyticity Condition

In this section, we will prove the following lemma.

Lemma 4.1. *Let M be a C^r-m-manifold embedded in $\mathbb{R}^n$ ($r \geq 1$ finite or infinite). Then there is a C^r-diffeomorphic analytic manifold M^* in $\mathbb{R}^n$.*

Once proved, we apply this lemma to the embedding of M in $\mathbb{R}^{2m+1}$ to prove Theorem 1.1.

For this we will need the following lemma.

Lemma 4.2. *Given an open set R in $\mathbb{R}^n$, a positive continuous function $\eta : R \to \mathbb{R}$, and $r \geq 0$, there is an analytic function $\omega : R \to \mathbb{R}$ such that*

$$\omega(q) > 0$$
$$|D_k \omega(q)| < \eta(q)$$

for $q \in R$ where D_k is the total kth derivative (where $k \leq r$)

Proof. Let $\{C_i\}$ be a locally finite open cover of R, and let $\phi_i : R \to \mathbb{R}$ be a C^∞ partition of unity subordinate to the $\{C_i\}$. Choosing $a_i \in \mathbb{R}^+$ such that $a_i < c_i$ where c_i is the minimum value of η on C_i, we have that $\phi : R \to \mathbb{R}$ given by $\phi(q) = \sum_{C_i} a_i \phi_i(q)$ is a positive C^∞ function where $\phi(q) = \sum a_i \phi_i(q) < \sum c_i \phi_i(q) \leq \eta(q)$. If $\omega : R \to \mathbb{R}$ is an analytic function approximating ϕ on R through the rth order within a small positive function $\gamma : R \to \mathbb{R}$ chosen such that $\gamma(q) > 0$ for all $q \in R$ and $|D_k \omega(q)| < \eta(q)$ for $k \leq r$, the existence of which is given by Lemma 3.1, then ω satisfies the Lemma.

□

4.1. **The Function ϕ^*.** Let M be a C^r-m-manifold embedded in $\mathbb{R}^n$. Now define P, R, ξ and H as in Lemma 3.3. We will extend the domains of P, R, and ξ from M to $R(M)$ by setting $P(q) = P(H(q))$ and $\xi(q) = \xi(H(q))$ for each $q \in R(M)$, noting that these functions remain continuous since H is C^r in $R(M)$. These extensions naturally give rise to an extension of R given by $R(q) = R(H(q))$.

We now define $\phi : R(M) \to \mathbb{R}$ by

$$\phi(q) = ||q - H(q)|| \tag{1}$$

which is the distance between a point in $q \in R(M)$ from the intersection of M and $P(q)$. Since H is C^r in $R(M)$, we get that ϕ is C^r in $R(M) \backslash M$ (Where ϕ is not necessarily C^r in M because the norm is not differentiable at 0. We do, however, have that ϕ is continuous on $R(M)$).

Let $\eta(q) = min(\frac{1}{3}, \frac{1}{3}\phi(q))$ for $q \in R(M)$. By Lemma 3.1, there exists a function ϕ' continuous in $R(M)$ and analytic in $R(M) \backslash M$ such that ϕ' approximates ϕ in $R(M) \backslash M$

within η through the first order. Then, by Lemma 3.2, we know that ϕ' is C^1 in $R(M)$ and $\phi' = \phi$ in M. By our definition of η, we know that

$$
\begin{aligned}
|\phi'(q) - \phi(q)| &< \frac{1}{3}\xi(q) \\
||\nabla\phi'(q) - \nabla\phi(q)|| &< \frac{1}{3}
\end{aligned}
\tag{2}
$$

in $R(M)\backslash M$, and that $\phi' = \phi = 0$ in M. Similarly, by Lemma 4.2, there is a positive analytic function $\omega : R(M) \to \mathbb{R}$ such that

$$
\begin{aligned}
|\omega(q)| &< \eta(q) \\
||\nabla\omega(q)|| &< \eta(q)
\end{aligned}
\tag{3}
$$

and hence

$$
\begin{aligned}
|\omega(q)| &< \frac{1}{3}\xi(q) \\
||\nabla\omega(q)|| &< \frac{1}{3}
\end{aligned}
\tag{4}
$$

for $q \in R(M)$. We will now define a third function $\phi^* : R(M) \to \mathbb{R}$ by

$$\phi^*(q) = \phi'(q) - \omega(q) \tag{5}$$

which is continuous on $R(M)$ and analytic on $R(M)\backslash M$. Furthermore, since $\phi'(p) = 0$ and $\omega(p) > 0$ for $p \in M$, we know that $\phi^*(p) < 0$ for $p \in M$. We now look to show that ϕ^* vanishes on some subset of $P(p)$ for each $p \in M$ (which we will show is a level set for $\phi^*(q) = 0$ that is diffeomorphic to an $(n-m-1)$-sphere), and then show this set of points is characterized by a function that varies analytically with $q \in R(M)$ and $Q(q)$ close to $P(q)$ and containing q. This function will be used to create an analytic neighborhood, the boundary of which is an $(n-1)$-manifold S, of M that we will use to define our analytic manifold M^*. We also necessitate that this neighborhood's center of mass lies on its interior, the reason for which will be made more clear later on.

4.2. **The Neighborhood S.** We begin by defining an analytic orthogonal transformation that sends an $(n-m)$-plane P to another such plane P' and sends a specific point p in P to a specific point p' in P'. Since our planes will be relatively close, and hence the transformation will be close to the identity matrix, we will assume that our transformation has an empty kernel.

4.2.1. *Transformation $T_{p,P}$.* Let P_0 be the plane parallel to P that passes through the origin, and let P_0' be the plane parallel to P' that passes through the origin. Let $v_1, \ldots, v_k$ be mutually orthogonal vectors that span P_0, and $v_{k+1}, \ldots, v_n$ be vectors orthogonal to each other as well as to $v_1, \ldots, v_k$ that span E_n. To each v_i, we define its image Tv_i explicitly and inductively. For $1 \leq i \leq k$, let $v_i' = proj(v_i, P_0')$ be the projections of v_i onto the plane P_0'. Let P_i' be the plane in P_0' formed by the span of $v_1', \ldots, v_i'$. We can now apply the Gram-Schmidt process to the vectors $v_1', \ldots, v_i'$ to create orthogonal vectors $Tv_1, \ldots, Tv_i$ which preserve the orientation of $v_1, \ldots, v_i$. For $j > k$, let P_j' be the plane spanned by P_0' and $v_{k+1}, \ldots, v_j$. We then apply Gram-Schmidt again to create vectors $Tv_1, \ldots, Tv_j$ that span this space and are orthogonal to each other while preserving the orientation of $v_1, \ldots, v_j$. We then define $T_{p,P}$ to be the composition of the translation sending p to 0, T as defined above, and the translation sending 0 to p', which is the unique transformation that sends p to p' and v_i to Tv_i for $i = 1, \ldots, n$.

4.2.2. *The Analytic Function σ.* In order to prove that S exists and show how it varies analytically with q and $Q(q)$, we will need to extend the definitions of some of our functions, and define a few new functions. Given a subset K of M, let $R(K)$ be the set of all $q \in R(M)$ such that $H(q) \in K$. Fix $p_0 \in M$. Let T_{p_0} be the plane tangent to M at p_0 contained in $\mathbb{R}^n$. Since M is analytic, if we choose a small enough neighborhood U of p_0, then we have that $P(p)$ is complementary to T_{p_0} for all $p \in U$. We can then further reduce our neighborhood to a $U' \subset U$ with an associated $\delta > 0$ such that if $Q(q)$ is any plane through some $q \in R(U')$ where $||Q(q) - P(q)|| < \delta$, then $Q(q)$ is complementary to T and hence it intersects T in a unique point $H^*(Q(q))$. Since H^* is simply the projection of points in a plane to a fixed point on a line, it is analytic. We now use H^* to define three new functions.

$$
\begin{aligned}
H'(q) &= H^*(P(q)) \\
u(q) &= \frac{q - H(q)}{||q - H(q)||} \\
u'(q) &= \frac{q - H'(q)}{||q - H'(q)||}
\end{aligned}
\tag{6}
$$

Here, $u(q)$ is a unit vector from q in the direction of it's projection onto M and $u'(q)$ is a unit vector from q in the direction of it's projection onto T. If U' and δ are sufficiently small, then for all $q \in R(U')$ and any $Q(q)$ passing through q such that $||Q(q) - P(q)|| < \delta$, we have that $\phi^*(H^*(Q(q))) < 0$, $||H'(q) - H(q)|| < \frac{\xi(q)}{6}$, and if $\phi^*(q) \geq 0$ then $||u'(q) - u(q)|| < \frac{1}{3}$.

Consider the function $T_{H^*(Q(q)),Q(q)}$ as defined in subsection 4.2.1 for any $Q(q)$ satisfying the above conditions (We will call the space of all such $Q(q)$ Y) and the fixed point p_0 and plane $P(p_0)$. As a reminder, this function analytically maps $P(p_0)$ onto $Q(q)$ where $T_{H^*(Q(q)),Q(q)}(p_0) = H^*(Q(q))$. Let $S(Q(q))$ be the unit $(n - m - 1)$-sphere in $Q(q)$ about $H^*(Q(q))$. For any $q_0 \in S(P(p_0))$, let correspond $\mu(Q(q), q_0) = T_{H^*(Q(q)),Q(q)}(q_0)$ in $S(Q(q))$. Since $T_{H^*(Q(q)),Q(q)}$ is analytic on the plane $P(p_0)$ and the Gram-Schmidt process is analytic, $\mu : Y \times S(P(p_0)) \to R(U')$ is analytic.

For each $q_0 \in S(P(p_0))$, $\alpha > 0$, and $Q(q)$, let

$$w = H^*(Q(q)) + \alpha(\mu(Q(q), q) - H^*(Q(q))) \tag{7}$$

be a point on the image of $S(P(p_0))$ dilated by α. We now define $\sigma : Y \times S(P(p_0)) \times \mathbb{R}^+ \to R(M)$ by

$$\sigma(Q(q), q_0, \alpha) = \phi^*(w) \tag{8}$$

which is analytic on $R(M)\backslash M$ since ϕ^*, μ, and H^* are analytic there.

4.2.3. *The zero set of* σ. It remains to be seen that the zero set of σ bounds an analytic $(n-1)$-neighborhood of M. That is, for some γ where $0 < \gamma < \delta$, letting Q be any plane through some $q \in R(U')$ where $||Q - P(q)|| < \gamma$, and for each $q_0 \in S(P(p_0))$, there is a unique

$$\alpha = \rho(Q, q_0) > 0 \tag{9}$$

determined by an analytic function $\rho : Y \times S(P(p_0)) \to \mathbb{R}^+$ which causes σ to vanish. First, by the Analytic Implicit Function Theorem, we know that ρ exists and is analytic as σ is analytic, provided that we can show $\frac{\partial\sigma}{\partial\alpha}\Big|_{(Q,q_0,\beta)} \neq 0$ whenever $\sigma(Q, q_0, \beta) = 0$. Furthermore, ϕ^* is both analytic and bounded, so it's zero set bounds an analytic neighborhood of M provided that $\sigma(Q, q_0, \rho(Q, q_0))$ traces a a shape diffeomorphic to a sphere containing $H^*(Q)$. Since $T_{H^*(Q),Q}$ is an orthogonal linear transformation, we know that μ sends the unit sphere $S(P(p_0))$ to a unit sphere in Q about $H^*(Q)$. Also, for every point on a line segment from $H^*(Q)$ to any point of $S(Q)$, we can find an α such that the point lies on a sphere of radius α about $H^*(Q)$ and therefore can be expressed in the form of w for some $q_0 \in S(P(p_0))$. So, it is sufficient to examine points on an arbitrary line segment from $H^*(Q)$ to a point of $S(Q)$ for an arbitrary choice of Q when characterizing σ. Therefore, it is sufficient for us to find a point q in the line segment from $H^*(Q)$ to a point q' of $S(Q)$ such that $\phi^*(q) = 0$ and $|\frac{\partial\sigma}{\partial\alpha}| > 0$ for all points w in the line from $H^*(Q)$ to q' such that $\phi^*(w) \geq 0$ (where this condition on the derivative ensures there are no critical points in the non-negative region, therefore guaranteeing that the uniqueness of the zero on the line as $\phi*$ must eventually increase to zero as it moves away from M, it must transition to being positive after that since the derivative is non-zero there, and it cannot decrease back to zero once it does since that would require a critical point in the positive region).

We will prove our result on planes of the form P as in Lemma 3.3 (noting that all $P(p)$ are degenerative members of Y). There is then some open neighborhood around q in which $|\frac{\partial\sigma}{\partial\alpha}| > 0$ and hence our result holds for any arbitrary plane that is sufficiently close to P, provided it is also within δ (We let γ be this distance). Since we will be examining a line in some P, we have that

$$\sigma'(p, q_0, \alpha) = \sigma(P(p), q_0, \alpha)$$

for $p \in U'$ is equivalent to σ on our region. Hence, they will be used interchangeably until we prove the existence and uniqueness of such a q.

To summarize, we aim to prove the following lemma.

Lemma 4.3. *Fix* $p \in U'$ *and* $q_0 \in S(P_0)$. *Then,*

- *There is a point q on the line segment between $H^*(P(p))$ and $\mu(P(p), q_0)$ such that $\phi^*(q) = 0$.*
- $|\frac{\partial \sigma'}{\partial \alpha}| > 0$ *for all points w in the line between $H^*(P(p))$ and $\mu(P(p), q_0)$ such that $\phi^*(w) \geq 0$.*

Proof. Fix $p \in U'$ and $q_0 \in S(P_0)$. By our definition of H', we know that $H^*(P(q)) = H'(q)$ and hence $||H^*(P(q)) - H(q)|| < \frac{\xi(q)}{6}$ for $q \in R(U')$. So, if w is the point on the line segment from $H^*(P(p))$ to $\mu(P(p), q_0)$ such that the corresponding $\alpha = \frac{5\xi(p)}{6}$, then the line segment from $H^*(P(p))$ to w lies entirely in $R(U')$. Furthermore, the following shows that $\phi^*(w) > 0$ (noting that $\xi(w) = \xi(p)$ since $H(w) = p$)

$$\begin{aligned}
\phi^*(w) &= \phi'(w) - \omega(w) \\
&> \phi(w) - \frac{1}{3}\xi(w) - \omega(w) \\
&> \phi(w) - \frac{1}{3}\xi(w) - \frac{1}{3}\xi(w) \\
&= \phi(w) - \frac{2}{3}\xi(w) \\
&= ||w - H(w)|| - \frac{2}{3}\xi(w) \\
&= ||w - p|| - \frac{2}{3}\xi(p) \\
&= ||w - H^*(P(p)) - (p - H^*(P(p)))|| - \frac{2}{3}\xi(p) \\
&\geq \big| ||w - H^*(P(p))|| - ||p - H^*(P(p)))|| \big| - \frac{2}{3}\xi(p) \\
&= \big| \alpha - ||p - H^*(P(p)))|| \big| - \frac{2}{3}\xi(p) \\
&= \alpha - ||p - H^*(P(p)))|| - \frac{2}{3}\xi(p) \\
&> \alpha - \frac{1}{6}\xi(p) - \frac{2}{3}\xi(p) = 0
\end{aligned}$$

But then, we know that $\phi^*(H^*(P(p))) < 0$ by our choice of U' and δ, and hence there is a point on the line segment from $H^*(P(p))$ to w such that $\phi^* = 0$ since ϕ^* is continuous.

Now that we have shown the existence of such a point, we will prove that it is unique. Consider now the Jacobian representation of $\nabla \phi$. In order to find $Proj_{P(p)} \nabla \phi(q)$ for $q \in P(p)$, we multiply on the left by the projection matrix onto $P(p)$, giving us $[Proj_{P(p)}][\nabla \phi][q]$. Evaluating $[Proj_{P(p)}][\nabla \phi]$ gives us $\nabla(\phi|_{P(p)})$ as defined as a function in $P(p)$. We then note that H is constant within $P(p)$. Therefore, we have

$$\begin{aligned}
Proj_{P(p)} \nabla\phi(q) &= \frac{d}{d\overline{q}} ||q-p||_{P(p)} \\
&= \frac{q-p}{||q-p||} * \frac{d}{d\overline{q}}(q-p) \\
&= \frac{q-p}{||q-p||} = u(q)
\end{aligned}$$

where $\overline{q}$ is the portion of q that lies inside of $P(p)$. Hence, $\nabla\phi(w') \cdot u' = u(w') \cdot u'$ for any vector u' parallel to $P(p)$.

Take any w' on the line segment from $H^*(P(p))$ to w for which $\phi^*(w') \geq 0$. Since $\mu(P(p), q_0)$ is on the unit sphere around $H^*(P(p))$ in the plane $P(p)$, we know $||\mu(P(p), q_0) - H^*(P(p))|| = 1$, $H'(\mu(P(p), q_0)) = H^*(P(\mu(P(p), q_0))) = H^*(P(p)) = H'(w')$, and $u'(\mu(P(p), q_0)) = u'(w')$. Then, using the same result about the projection of the gradient as above,

$$\begin{aligned}
\frac{\partial\sigma'}{\partial\alpha}(w') &= \nabla\phi^*(w') \cdot \frac{\mu(P(p), q_0) - H^*(P(p))}{||\mu(P(p), q_0) - H^*(P(p))||} \\
&= \nabla\phi^*(w') \cdot \frac{\mu(P(p), q_0) - H'(w')}{||\mu(P(p), q_0) - H'(w')||} \\
&= \nabla\phi^*(w') \cdot u'(\mu(P(p), q_0)) \\
&= \nabla\phi^*(w') \cdot u'(w')
\end{aligned}$$

This tells us that

$$\begin{aligned}
\left|\frac{\partial\sigma'}{\partial\alpha}(w')\right| &= |\nabla\phi^*(w') \cdot u'(w')| \\
&= |(\nabla\phi^*(w') - \nabla\phi(w')) \cdot u'(w') + \nabla\phi(w') \cdot u'(w')| \\
&= |(\nabla\phi^*(w') - \nabla\phi(w')) \cdot u'(w') + u(w') \cdot u'(w')| \\
&= |(\nabla\phi^*(w') - \nabla\phi(w')) \cdot u'(w') + u(w') \cdot (u'(w') - u(w')) + u(w') \cdot u(w')| \\
&= |(\nabla\phi^*(w') - \nabla\phi(w')) \cdot u'(w') + u(w') \cdot (u'(w') - u(w')) + 1| \\
&= |(\nabla\phi'(w') - \nabla\phi(w') - \nabla\omega(w')) \cdot u'(w') + u(w') \cdot (u'(w') - u(w')) + 1| \\
&= |(\nabla\phi'(w') - \nabla\phi(w')) \cdot u'(w') - \nabla\omega(w') \cdot u'(w') + (u'(w') - u(w')) \cdot u(w') + 1| \\
&\geq -|(\nabla\phi'(w') - \nabla\phi(w')) \cdot u'(w')| - |\nabla\omega(w') \cdot u'(w')| - |(u'(w') - u(w')) \cdot u(w')| + 1 \\
&\geq -|\nabla\phi'(w') - \nabla\phi(w')| - |\nabla\omega(w')| - |u'(w') - u(w')| + 1 \\
&> \frac{-1}{3} - \frac{1}{3} - \frac{1}{3} + 1 = 0
\end{aligned}$$

since $|u(q)| = |u'(q)| = 1$, $||u'(q) - u(q)|| < \frac{1}{3}$, $||\nabla\phi'(q) - \nabla\phi(q)|| < \frac{1}{3}$, and $||\nabla\omega(q)|| < \frac{1}{3}$ for all $q \in R(U')$. $\square$

By our previous logic, we have proved that ρ is analytic.

We now take any $q \in R(U')$ and $Q(q)$ such that $||Q(q) - P(q)|| < \gamma$. Since $Q(q)$ is fixed, $\sigma(Q(q), q_0, \rho(Q(q), q_0))$ is a function of q_0 that maps $S(P(p_0))$ into the intersection of the zero set of ϕ^* and $Q(q)$, which we will call $S^*(Q(q))$. As we have proven, this is the boundary of an analytic neighborhood of $H^*(Q(q))$. Since σ is analytic, and in particular analytic in $Q(q)$, $S^*(Q(q))$ varies analytically with our choice of $Q(q)$. Since our manifold M is second-countable, the open cover of $\bigcup_{p \in M} U'_p$ where U'_p is the neighborhood U' where $p_0 = p$ has a countable subcover with a locally finite refinement. Each of these sets has an associated γ_p that allows for a function ρ in that neighborhood. We then let $\gamma : R(M) \to \mathbb{R}^+$ be a continuous function satisfying $\gamma(q) = \gamma(H(q))$ is less than each γ_p associated with each neighborhood U' that contains $H(q)$. Now if q is any point of $R(M)$ and P^* is a function (that defines planes that fill out a neighborhood of M in a one to one way in the same way as P) containing q and satisfying

$$||P^*(q) - P(q)|| < \gamma(q) \tag{10}$$

$R(M)$ intersects S in sets $S^*(q, P^*(q))$ diffeomorphic to a sphere which varies analytically with q and P^*.

4.3. **The Analytic Manifold** M^***.** For any $q \in R(M)$ and plane $Q(q)$ satisfying (10), let $Q^*(q, Q(q))$ be the interior of $S^*(q, Q(q))$. We then define $g(q, Q(q))$ to be the center of mass of $Q^*(q, Q(q))$. We now look to show that the collection M^* of all points $q = g(q, P^*(q))$ where P^* is a sufficiently close analytic approximation of P (satisfying (10)) in $R(M)$ is an analytic manifold embedded in $R(M)$ that is C^r-homeomorphic with M, completing the proof.

First, we must show that g is analytic. For any $q \in R(M)$ and $Q(q)$ satisfying (10), q lies in some U' from the open cover described above. We then associate with it some $p_0 \in M$ of which U' is a neighborhood. If $V(q, Q(q))$ is the $(n-m)$-volume of $Q^*(q, Q(q))$ with volume element $|dq|$, then we can express g as

$$\begin{aligned}
g(q, Q(q)) &= \frac{1}{V(q, Q(q))} \int_{Q^*(q,Q(q))} \overline{q} |d\overline{q}| \\
&= \frac{\int_{Q^*(q,Q(q))} \overline{q} |d\overline{q}|}{\int_{Q^*(q,Q(q))} 1 |d\overline{q}|} \\
&= \frac{\int_{S(P(p_0))} \int_0^{\rho(q',Q(q))} \alpha^{n-m-1} (H^*(Q(q)) + \alpha(\mu(Q(q), q') - H^*(Q(q)))) d\alpha |dq'|}{\int_{S(P(p_0))} \int_0^{\rho(q',Q(q))} \alpha^{n-m-1} d\alpha |dq'|} \\
&= \frac{\int_{S(P(p_0))} \int_0^{\rho(q',Q(q))} \alpha^{n-m-1} (H^*(Q(q)) + \alpha(\mu(Q(q), q') - H^*(Q(q)))) d\alpha |dq'|}{\int_{S(P(p_0))} \frac{\rho(q',Q(q))^{n-m}}{n-m} |dq'|}
\end{aligned}$$

Since ρ is positive everywhere, we know that $\int_{S(P(p_0))} \frac{\rho(q',Q(q))^{n-m}}{n-m} |dq'|$ is never zero. Therefore, we can note that g is a ratio of compositions of analytic functions and hence is analytic.

We now aim to characterize our analytic manifold. Fix U' and $p_0 \in M$ associated with some neighborhood U'. We then define $\tau : R(U') \times Y \to P(p_0)$ by

$$\tau(q, Q) = T^{-1}_{H^*(Q(q)),Q(q)}(q) - T^{-1}_{H^*(Q(q)),Q(q)}(g(q, Q(q)))$$

We then subsequently define $\tau'(q) = \tau(q, P(q))$ and $\tau^*(q) = \tau(q, P^*(q))$ where P^* analytically approximates P within γ through the first order. Since $P(p_0) \cong \mathbb{R}^n$, τ' is then a smooth map that maps each $P(q)$ diffeomorphically onto $\mathbb{R}^{n-m}$. Similarly, τ^* is an analytic map of each $P^*(q)$ analytically-diffeomorphically onto $\mathbb{R}^{n-m}$. We then let $M^*_{U'}$ be the set of all $q \in R(U')$ such that $\tau^*(q) = 0$ (equivalently, this is the image of 0 under $(\tau^*)^{-1}$). Fix $q \in R(U')$. We then let $v_1, \ldots, v_{n-m}$ be orthogonal vectors spanning $P^*(q)$ and $v_{n-m+1}, \ldots, v_n$ be vectors orthogonal to each other and to $v_1, \ldots, v_{n-m}$ (These can be found in a process similar to section 4.2.1). By the Analytic Implicit Function Theorem, letting q_m be the last m coordinates of q in the coordinate system induced by $v_1, \ldots, v_n$, we can find an open set $A \subset \mathbb{R}^m$ around each q_m where we have a function $b : A \to B \cong R^{n-m}$ where B is the tangent space to M such that $\tau^*(b(q_m), q_m) = 0$ for all $q_m \in A$. $q_m \to (b(q_m), q_m)$ is then a chart, and hence this level set is an analytic submanifold of $R(U')$. Also, the kernel of $D\tau^*(q)$ at any $q \in M^*_{U'}$ is precisely the tangent space to $M^*_{U'}$ at q; this is complementary to $P^*(q)$, and since P^* is a close approximation to P through the first order, this is also complimentary to $P(q)$.

We now apply this to each U' covering M and find a collection of analytic manifolds $M^*_{U'}$. Since the center of mass g is dependant on $q, Q(q)$ and not U', the union of these manifolds M^* is an analytic manifold that is locally equal to each $M^*_{U'}$. We then restrict the projection H to M^* to define a smooth map $M^* \to M$ that is one to one and whose derivative sends each tangent space at $q \in M^*$ onto the tangent space at $H(q) \in M$. By the Inverse Function Theorem, this is a C^r-diffeomorphism.

5. Bibliography

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